AGGREGATIVE CONVEXITY AND THE EXISTENCE OF COMPETITIVE EQUILIBRIUM

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1. INTRODUCTION

The conditions for the Walrasian system of general equilibrium to be consistent are investigated first by Wald [9, 10] and more recently by Arrow and Debreu [1], Gale [3], McKenzie [6, 7], and Nikaidô [8]. The Wald model postulates the continuity of demand functions and the linearity of production structures, while in the Arrow-Debreu model the existence of a competitive equilibrium is asserted under conditions which are imposed on individual economic units. More specifically, it has been shown in Arrow and Debreu [1] that if each consumer has a continuous, convex, and nonsaturated preference, each firm's production possibility set is closed, convex, and the aggregate production possibility set is irreversible, and if initial holdings are sufficiently abundant, then there always exists a competitive equilibrium. In the present note, we are interested in showing that Arrow and Debreu's existence theorems remain valid under slightly weaker conditions than those postulated in [1], and a simplified proof for the second existence theorem will be presented.

The condition with which we are particularly concerned here is the one which postulates the convexity of each firm's production possibility set; we shall instead postulate that the aggregate production possibility set is convex (and closed). In the case of each firm's production possibility set, it is by no means justifiable to assume the absence of those factors such as indivisibility or uncertainty which cause the non-convexity. As for the aggregate production possibility set, however, the convexity is verified under fairly weak conditions on production processes of each individual firm. In particular, it is shown that as the number of firms increases, the aggregate production possibility set tends to be convex, or the non-convexity part becomes negligible, whatever the shape of each firm's production possibility set may be. One of the institution hypotheses underlying the model of competitive economy is that there exists a large number of firms, none of which dominates the market, and the convexity of the aggregate produc-

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2) The asymptotic convexity of the aggregate production possibility set was shown by Farrell [2] for the case in which every firm has an identical production possibility set, and by Hurwicz and Uzawa [4] for the general case.
tion possibility set may be justified as one of the postulates for a model of competitive
equilibrium.

2. THE ARROW-DEBREU MODEL OF COMPETITIVE EQUILIBRIUM

We assume that there exist distinct commodities of a finite number, say \( n \); commodities are labeled \( i=1, \ldots, n \). Commodity bundles are represented by vectors with \( n \)-components, the \( i \)-th component being the quantity of commodity \( i \) to be produced or consumed. The Arrow-Debreu model of competitive equilibrium consists of consumption units and production units of finite numbers, to be labeled \( r=1, \ldots, R \), and \( s=1, \ldots, S \), respectively. Let \( X_r \) be the (conceivably) available set of commodity bundles for consumption unit \( r \), and \( u_r(x_r) \) the utility function which represents the preference of \( r \), \( r=1, \ldots, R \). It will be assumed that:

I. For each consumption unit \( r \), \( X_r \) is a closed, convex, non-empty set of commodity bundles, and is bounded from below.

II. For each consumption unit \( r \), \( u_r(x_r) \) is defined, continuous, non-saturated, and strongly quasi-concave on \( X_r \).

For each production unit \( s \), the feasible production processes are described by specifying the production possibility sets \( Y_s \); the aggregate production possibility set \( Y \) is defined as the sum of individuals' production possibility sets:

\[ Y = Y_1 + \ldots + Y_s. \]

III. For each production unit \( s \), the production possibility set \( Y_s \) contains 0.

IV. The aggregate production possibility set \( Y \) is closed, convex, and irreversible.\(^3\)

Let \( \alpha_r(\rho) \) be the function which describes the portion of the profit of production units to be shared by consumption unit \( r \). It will be assumed that:

V. For any \( r \) and \( s \), \( \alpha_r(\rho) \) is continuous, non-negative, and

\[ \sum_{r=1}^{R} \alpha_r(\rho) = 1. \]

Finally, the commodity bundles initially held by consumption units are denoted by \( \zeta_1, \ldots, \zeta_R \).

A state of the economy is described by specifying commodity bundles \( x_1, \ldots, x_R \) to be consumed by consumption units, commodity bundles \( y_1, \ldots, y_S \) to be produced by production units, and a vector of prices \( p \) prevailing in the economy.

A state \( (x_1^*, \ldots, x_R^*, y_1^*, \ldots, y_S^*, p^*) \) is defined as a competitive equilibrium with respect to the initial holdings if

(i) \( p^* \geq 0 \);

(ii) for any consumption unit \( r \),

\[ x_r^* \in X_r, \]
\[ u_r(x_r^*) = \max \{ u_r(x_r) ; x_r \in X_r, p^* \cdot x_r^* \leq M_r^* \}; \]

where

\[ 3) \text{ A set } Y \text{ here is termed irreversible if (a) } y \geq 0, y \in Y, \text{ imply } y=0, \text{ and (b) } Y \cap (-Y) = 0. \]
(iii) For any production unit $s$,
\[ x_s \in Y_s , \]
\[ p^* \cdot y_s^* = \max_{y_s \in Y_s} p^* \cdot y_s ; \]
(iv) $z^* \leq 0$, $p^* \cdot z^* = 0$,
where
\[ z^* = x^* - y^* - \zeta , \]
\[ x^* = \sum_{r=1}^{R} x_r^* , \quad y^* = \sum_{s=1}^{S} y_s^* , \quad \zeta = \sum_{r=1}^{R} \zeta_r . \]

3. **ARROW AND DEBREU'S FIRST EXISTENCE THEOREM**

**Arrow and Debreu's First Existence Theorem:** Any model satisfying assumptions I–IV has a competitive equilibrium if there are $x_r \in X_r$ such that $\zeta_r > x_r$, $r=1, \ldots, R$.

**Proof:** For any set $A$ of commodity bundles, let the smallest closed, convex set containing $A$ be denoted $[A]$. Then we have from (1) that
\[ [Y_1] + \ldots + [Y_s] = [Y] . \]
Since $Y$ is closed and convex by (IV),
\[ [Y] = Y ; \]
hence,
\[ [Y_1] + \ldots + [Y_s] - Y , \]
where $[Y_s]$ are all closed and convex.

From (2) and the definition of competitive equilibria, it can be shown that if there exists a competitive equilibrium for the model $(X_r, [Y_s])$, there also exists a competitive equilibrium for the model $(X_r, Y_s)$. Thus it suffices to show that the model $(X_r, [Y_s])$, where each production possibility set $[Y_s]$ is closed and convex, has a competitive equilibrium. Changing the notation, we may from the beginning assume that for each $s$, the production possibility set $Y_s$ is closed and convex.

By applying an argument similar to the one in Arrow and Debreu ([1], pp. 276–77), we may also without loss of generality assume that, for each $r$ and $s$, $X_r$ and $Y_s$ are compact (=closed and bounded).

Let $P$ be the set of all normalized price vectors, i.e.,
\[ P = \{ p = (p_1, \ldots, p_n) : p \geq 0 , \quad \sum_{i=1}^{n} p_i = 1 \} . \]
For any non-empty compact set $B$, let the function $\rho_B$ on $P$ be defined by
\[ \rho_B(p) = \sup_{y_B} p \cdot y . \]
Since $B$ is compact, we have
\[ \rho_B(p) = \max_{y_B} p \cdot y . \]
and $\rho_B$ is a continuous function of $p$. 

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Let $A$ be a non-empty compact convex set, and $u$ be a function on $A$. For any $p \in P$ and any number $M$ such that

$$M \geq \mu(p) = \min_{x \in A} p \cdot x,$$

the set

$$\{ x : x \in A, p \cdot x \leq M \}$$

is non-empty, compact, and convex.

Now we define a vector $x^*$ such that $x^* \in A$, $p \cdot x^* \leq M$, and $u(x^*) \leq u(x)$ for all $x \in A$ fulfilling $p \cdot x \leq M$.

$x^*$ is uniquely determined by $(p, M)$ and will be denoted by $x^* = \phi_A, u(p, M)$. We shall call function $\phi_A, u$ the demand function associated with a set $X$ and a function $u$.

It is easily seen that $\phi_A, u$ is continuous at $(p, M)$ for which $M > \mu(p)$.

Let $B$ be a non-empty compact set, and $c$ a vector such that

$$c > 0.$$  

For any vector $x$ such that there are no vectors $y \in B$ with $x < y$, we shall define a number $\beta(x)$ as the infimum of non-negative numbers $\beta$, for which there is a vector $y \in B$ such that

$$y \geq x - \beta c.$$  

In other words,

(3)  \[ \beta(x) = \inf \{ \beta : \beta \geq 0, \, \Pi_{x-\beta c} \cap B \neq \phi \} , \]

where

(4)  \[ \Pi_{x-\beta c} = \{ u ; u \in \mathbb{R}, \, u \geq x - \beta c \} , \]

and $\phi$ denotes the empty set.

Since $Y$ is compact and $c > 0$,

(5)  \[ 0 \leq \beta(x) < \infty . \]

The function $\beta$ thus defined will be denoted by $\beta_{B, c}$. Now we shall show that $\beta = \beta(x)$ if and only if

(6)  \[ \begin{cases} \Pi_{x-\beta(x)c} \cap B \neq \phi \\ \Pi_{x-\beta(x)c} \cap B \text{ lies on the boundary of } \Pi_{x-\beta(x)c} \end{cases} . \]

Since $B$ is compact, it is obvious that

$$\Pi_{x-\beta(x)c} \cap B \neq \phi.$$  

If $\Pi_{x-\beta(x)c} \cap B$ has an inner point $y$ of $\Pi_{x-\beta(x)c}$, then

$$y \in \Pi, \, y > x - \beta(x)c .$$  

Hence, there would be a number, $\beta$, such that

$$\beta < \beta(x) ,$$  

$$y > x - \beta c ,$$

which contradicts (3).

On the other hand, let $\beta$ be a number satisfying (6). It is obvious that
We shall show that the assumption
\[ \beta > \beta(x) \]
would lead to a contradiction.

Let \( y \in B \) be such that
\[ y > x - \beta(x)c \]
Then
\[ y > x - \beta c \]
which shows that \( y \) is contained in the interior of \( \Pi_{x-\beta c} \), thus contradicting (6).

We next show that \( \beta(x) \) is a continuous function of \( x \). To see this, let \( \{ x^\nu ; \nu = 1, 2, \ldots \} \) be a sequence such that
\[ \lim_{\nu \to \infty} x^\nu = x^0 \]
Then there are vectors \( y^\nu \in B \) such that
\[ y^\nu \geq x^\nu - \beta^\nu c \]
where
\[ \beta^\nu = \beta(x^\nu), \quad \nu = 1, 2, \ldots \]
Let \( \bar{\beta} \) be any accumulation point of the sequence \( \{ \beta^\nu \} \), and \( \{ \beta^{\nu*} \} \) be a subsequence of \( \{ \beta^\nu \} \) converging to \( \bar{\beta} \). Since \( B \) is compact, there is a sequence \( \{ y^{\nu*} \} \) such that
\[ \lim_{\nu \to \infty} y^{\nu*} = \bar{y} \in B \]
then
\[ \bar{y} \geq \bar{x} - \bar{\beta} c \]
which means
\[ \Pi_{x - \bar{\beta} c} \cap B \neq \emptyset \]
On the other hand, we shall show that \( \Pi_{x - \bar{\beta} c} \cap Y \) lies on the boundary of \( \Pi_{x - \beta c} \). Otherwise there would be a vector \( y \in Y \) such that
\[ y > x^0 - \bar{\beta} c \]
There is an integer \( k \) such that
\[ y > x^k - \beta^k c \]
which contradicts the definition of \( \beta^k \). Therefore
\[ \bar{\beta} = \beta(x^0) \]
for any accumulation point \( \bar{\beta} \) of \( \beta^\nu \), so that
\[ \lim_{\nu \to \infty} \beta(x^\nu) = \beta(x^0) \]
Now define a set \( A(x) \) of price vectors by
\[ A(x) = \{ q : q \in P, q \cdot y \leq \rho \cdot [x - \beta(x)c] \text{ for all } y \in B \} \]
Then \( A(x) \) is a non-empty, compact convex subset of \( P \).
Since \( B \) is a compact convex set, and \( \Pi_{x - \beta(c)x} \cap B \) lies on the boundary of \( \Pi_{x - \beta(x)c} \), there is a vector \( \bar{q} \) such that

\[ \bar{q} \cdot y \leq \bar{q} \cdot u \]

for all \( y \in B \) and \( u \in \Pi_{a \in \Delta(x)} \). Hence, \( \bar{q} \geq 0 \).

Let \( q = \frac{1}{\sum_{k} q_{k}^2} \), then \( \bar{q} \Delta(x) \). Convexity and compactness easily follow from (7).

We are now in a position to prove the theorem. For any \( p \in P \), let us define:

(8) \[ \rho_{s}(p) = \max_{r \in Y} \rho_{s}(p) \]

(9) \[ \rho(p) = \max_{r \in Y} \rho_{s}(p) \]

(10) \[ M_{r}(p) = p \cdot \zeta + \sum_{i=1}^{s} \alpha_{r}(\rho_{s}(p)) \cdot \rho_{s}(p) \]

(11) \[ x_{r}(p) = \phi_{x_{r}, y_{r}}(p, M_{r}(p)) \]

(12) \[ x(p) = \sum_{r=1}^{n} x_{r}(p) \]

(13) \[ \bar{\beta}(p) = \beta(x(p) - \zeta) \]

(14) \[ \Delta(p) = \Delta[x(p) - \zeta] = \{ q : q \in P, \rho(q) \leq \rho[x(p) - \zeta - \bar{\beta}(p) c] \} \]

Then

(15) \[ \rho(p) = \sum_{i=1}^{s} \rho_{s}(p) \]

(16) \[ p \cdot x(p) \leq M_{r}(p) \]

(17) \[ p \cdot x(p) \leq p \cdot \zeta + \rho(p) \]

Now we shall show that the function \( \beta(p) \) is upper semi-continuous at any \( p \in P \), in the sense that if

\[ \lim_{\nu \to -} p_{\nu} = p^{0}, \quad p^{0} \in P \]

\[ \lim_{\nu \to -} q_{\nu} = q^{0}, \quad q^{0} \in \beta(p^{0}) \]

then

\[ q^{0} \in \beta(p^{0}) \]

By the definition of \( \beta \),

\[ \rho(q_{\nu}) \leq q_{\nu} \cdot (x(p^{\nu}) - \zeta - \bar{\beta}(p^{\nu}) c), \nu = 1, 2, \ldots \]

Since \( \rho, x \) and \( \bar{\beta} \) are continuous,

\[ (q^{0}) \leq q^{0} \cdot (x(p^{0}) - \zeta - \bar{\beta}(p^{0}) c) \]

which means \( q^{0} \in \beta(p^{0}) \).

\( \beta(p) \) is a non-empty compact convex subset of \( P \) for any \( p \in P \), and \( \beta \) is upper semi-continuous, so that we can apply the fixed-point theorem of Kakutani [5] to \( \beta \). Consequently, there is a vector \( p^{*} \in P \) such that

\[ p^{*} \in \beta(p^{*}) \]

Let \( x_{r}^{*}, y_{r}^{*} \) be vectors such that
(18) \( x_r^* = x_r(p^*), \ r = 1, \ldots, R \) 

(19) \( x^* = \sum_{r=1}^{R} x_r^* \) 

(20) \( y^* \in \Pi x^* \cap Y \) 

(21) \( y^* = \sum_{i=1}^{S} y_i^*, \ y_s^* \in Y, \ s = 1, \ldots, S \) 

Then

(22) \( y^* \in Y \) 

(23) \( y^* \geq x^* - \zeta - \tilde{\beta}(p^*)c \) 

Hence,

(24) \( p^* \cdot y^* = \rho(p^*) = p^* \cdot (x^* - \zeta - \tilde{\beta}(p^*)c) \) .

Therefore, by (17),

(25) \( p^\cdot y^* \leq p^\cdot x^* - \tilde{\beta}(p^*)p^\cdot c \) .

Since \( c > 0 \) \( \tilde{\beta}(p^*) \geq 0 \) and \( p^* \in P \), we have

The relations (18–25) show that the state \( (x^*, y^*, p^*) \) is a competitive equilibrium.

Q.E.D.

4. ARROW AND DEBREU’S SECOND EXISTENCE THEOREM

VI. There are \( x, x \in X \), and \( y \in Y \) such that \( x < y + \zeta \), where \( x = \sum_{r=1}^{R} x_r \) .

A commodity \( h \) is said to be desirable if, for any vector \( x_r \in X_r, \ r = 1, \ldots, R \), there is a number \( \lambda > 0 \) such that

\[ x_r + \lambda \delta^h \in X \] 

\[ u_r(x_r + \lambda \delta^h) > u_r(x) \] 

where \( \delta^h = (0, \ldots, 1, \ldots, 0) \) .

The set of all desirable commodities is denoted by \( D \) .

VII. \( D \) is not empty.

A commodity \( h \) is said to be productive if, for any \( y \in Y \), there is a vector \( y_\infty \in Y \) such that

\[ y_{h'} \geq y_{h''} \] for all \( h' \neq h \) ,

\[ y_{h''} > y_{h'} \] for at least one \( h'' \in D \) .

The set of all productive commodities is denoted by \( P \) .

Arrow and Debreu’s Second Existence Theorem: Any model satisfying I—VI has a competitive equilibrium for any initial holding \( \zeta_1, \ldots, \zeta_R \) for which there are vectors \( x_r \in X_r \) such that

\[ x_r' \leq \zeta_r \] 

\[ x_r, h < \zeta_r \] for at least one \( h \in P \cup D \) ,

\( r = 1, \ldots, R \) .

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Proof: Let \((\xi_1', \ldots, \xi_R')\) be a sequence such that

\[
\zeta_r' > \xi_r', \quad r = 1, 2, \ldots,
\]

\[
\lim_{r \to \infty} \zeta_r' = \zeta_r, \quad \text{for all}\ r = 1, \ldots, R.
\]

Since \((\xi_1', \ldots, \xi_R')\) satisfies the assumption of the first existence theorem, there is a competitive equilibrium \((x_r', y_r', p')\) for \((\xi_1', \ldots, \xi_R')\):

\[
x_r' \in X_r, \quad y_r' \in Y_r, \quad p' \in P,
\]

\[
x_r' = \phi_r(p', M_r(p')),
\]

\[
p' \cdot y_r' = \rho_r(p'),
\]

\[
x_r' - \xi_r' \leq y_r',
\]

\[
p'(x_r' - \xi_r') = p' \cdot y_r'.
\]

where \(x_r' = \sum_{r=1}^{R} x_r', y_r' = \sum_{r=1}^{R} y_r', \xi_r' = \sum_{r=1}^{R} \xi_r'\).

The states \((x_r', y_r', p')\) are contained in compact sets, so that there is a subsequence \((x_r'^{k}, y_r'^{k}, p'^{k})\) such that

\[
\lim_{k \to \infty} x_r'^{k} = \bar{x}_r \in X_r,
\]

\[
\lim_{k \to \infty} y_r'^{k} = \bar{y}_r \in Y_r,
\]

\[
\lim_{k \to \infty} p'^{k} = \bar{p} \in P.
\]

Then

\[
(26) \quad \bar{p} \cdot \bar{y}_r = \rho_r(\bar{p})
\]

\[
(27) \quad \bar{p} \cdot \bar{x}_r \leq \bar{p} \cdot \xi_r + \sum_{i=1}^{S} \alpha_i \rho_i(\bar{p}) = M_r(\bar{p}),
\]

\[
\bar{x} - \xi \leq \bar{y},
\]

\[
\bar{p}(\bar{x} - \xi) = \bar{p} \cdot \bar{y},
\]

where

\[
\bar{x} = \sum_{r=1}^{R} \bar{x}_r, \quad \bar{y} = \sum_{r=1}^{R} \bar{y}_r.
\]

If \(\bar{p}_r > 0\) for all \(h \in P\), then \(M_r(\bar{p}) > \mu x_r(\bar{p}), \quad r = 1, \ldots, R\). Consequently, \(\phi_r(p, M_r(p))\) is continuous at \((\bar{p}, M_r(\bar{p}))\), so that

\[
\bar{x}_r = \phi_r(p, M_r(\bar{p})), \quad r = 1, \ldots, R.
\]

Therefore, \((\bar{x}_r, \bar{y}_r, \bar{p})\) is a competitive equilibrium for \((\xi_1', \ldots, \xi_R')\).

Q.E.D.

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